

The quark mass gap in a magnetic field

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Abstract

A magnetic field and the resulting Landau degeneracy enhance the infrared contributions to the quark mass gap. The gap does not grow arbitrarily, however, for models of asymptotic free interactions. For $B \rightarrow \infty$, the magnetic field decouples from the dimensionally reduced self-consistent equations, so that the gap behaves as $\sim \Lambda_{\text{QCD}}$ (or less), instead of $\sim \sqrt{|eB|}$. On the other hand, the number of participants to the chiral condensate keeps increasing as $\sim |eB|$ so that $|\langle \bar{\psi}\psi \rangle| \sim |eB|\Lambda_{\text{QCD}}$. After the mass gap stops developing, nothing tempers the growth of screening effects as $B \rightarrow \infty$. These features are utilized to interpret the reduction of critical temperatures for the chiral and deconfinement phase transitions at finite B , recently found on the lattice. The structures of mesons are analyzed and light mesons are identified. Applications for cold, dense quark matter are also briefly discussed.

1. Introduction

In past decades, systems in a magnetic field (B) have been useful laboratories to test theoretical ideas. A famous example is a system of cold atoms, in which a magnetic field controls the strength of the interactions. In QCD, similar utilities are also expected for the lattice Monte Carlo simulation at finite B [1–4]. In particular, we can study the nonperturbative gluon dynamics through polarization effects, controlling quark dynamics by a magnetic field. Such information may help the studies of cold, dense quark matter [5].

It seems that lattice studies already confirmed some of the theoretical ideas. At $T = 0$, a magnetic field enhances the size of the chiral condensate due to magnetic catalysis [6, 7]. A key feature of this phenomenon is the effective dimensional reduction. For $B \neq 0$, the phase space for the low energy particles and anti-particles is $\sim |eB| \int dp_{\parallel}$, increasing the number of participants to the formation of the chiral condensate. This should be contrast to the $B = 0$ case, where phase

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space quickly decreases as $\sim \int |p|^2 d|p|$ in the infrared region. In this case, due to the small number of participants, the system needs sufficiently strong attractive forces to form chiral condensates.

On the other hand, some surprises have been provided as well [4]. While B increases the chiral condensate below the (pseudo-)critical temperatures for the chiral restoration (T_χ) and deconfinement (T_D), those temperatures themselves decrease. This might contradict with our intuitions, if we think that a larger chiral condensate should generate a greater quark mass gap. Such thinking would suggest that (i) a larger quark mass gap should suppress thermal quark fluctuations, leading to increasing T_χ , and (ii) a larger quark mass gap suppresses quark loops, so that the results should approach to the pure gauge results, leading to increasing T_D .

To resolve this apparent contradiction, we shall argue that the quark mass gap at $T = 0$ can stay around $\sim \Lambda_{\text{QCD}}$ (or less) for large B . Then we can imagine that the gap at $T < T_{\chi,D}$ also stays around $\sim \Lambda_{\text{QCD}}$, without strongly suppressing thermal quark fluctuations and quark loops. If this is the case, the decreasing of critical temperatures would not be so unnatural.

In addition, the aforementioned behavior of the quark mass gap does not contradict with the growing behavior of the chiral condensate, but instead naturally explains its B -dependence at $T = 0$. In fact, the lattice results in [4] showed the behavior $\langle \bar{\psi}\psi \rangle_{T=0}^B \sim |eB|\Lambda_{\text{QCD}}$, for $|eB| \geq 0.3 \text{ GeV}^2 \gg \Lambda_{\text{QCD}}^2 (\simeq 0.04 \text{ GeV}^2)$. Noting that the relation under the Landau quantization,

$$\langle \bar{\psi}\psi \rangle_{4D} \sim |eB| \times \langle \bar{\psi}\psi \rangle_{2D}, \quad (1)$$

we can see that $\langle \bar{\psi}\psi \rangle_{2D}$ or the quark mass gap must be nearly B -independent and $O(\Lambda_{\text{QCD}})$.

In this work we will carry out all the calculations in the large N_c limit¹. The use of the large N_c is motivated by at least three reasons: (i) At large N_c , gluons are not screened, so the *nonperturbative* forces (i.e. the forces in the infrared) are stronger than the $N_c = 3$ case. Such forces can be used to set the *upper bound* of the quark mass gap. (ii) The large N_c limit has captured many qualitative aspects of the confined phase at $B = 0$. Therefore it is worth thinking and testing this approximation in the confined phase at finite B , since its validity and invalidity are not evident *a priori*. (iii) It is easy to imagine how the $1/N_c$ corrections *qualitatively* modify the large N_c results, and such corrections just provide welcomed effects for our scenario (see below).

We will use the large N_c limit to just claim that the quark mass gap does not grow much beyond Λ_{QCD} . To explain the reduction of the critical temperatures, in addition we have to argue the $1/N_c$ corrections. The quark loops as the $1/N_c$ corrections screen the nonperturbative forces. As B increases, the screening effects become larger because more low energy particles can participate to the gluon polarization, due to the enhanced Landau degeneracy $\sim |eB|$ in the lowest Landau

¹The large N_c limit in a magnetic field was also studied in Ref. [9] from a different perspective from ours.

level (LLL) [8]. If the quark mass gap stops growing as suggested in our scenario, there is nothing to suppress the growth of the screening effects as B increases². Therefore the nonperturbative forces are reduced at large B , and such reduction should lower the critical temperatures for given B . In addition, hadronic fluctuations as the $1/N_c$ corrections also grow as B increases, helping the chiral symmetry to restore [10]³.

We will argue that the demanded (nearly) B -independent gap of $O(\Lambda_{\text{QCD}})$ can be derived, provided that it is dominantly created by the nonperturbative force mediated by the IR gluons. In particular, both the IR enhancement (that is more drastic than the perturbative $1/p^2$ case) and the UV suppression of the gluon exchanges are crucial for our discussions. Since we will deal with the LLL which is essentially soft physics in the present paper, IR enhanced gluon is a key feature in this study (see [12] for a review). If we include only the perturbative $1/p^2$ force, the gap is much smaller than Λ_{QCD} and depends on B at most logarithmically. Similar arguments have been used in studies of the quark mass function at finite quark density [13, 14].

To illustrate our points, we first consider the NJL model which does not have the abovementioned properties. For $|eB| \rightarrow \infty$, the gap equation within the LLL approximation is

$$M_{\text{NJL}}(B) = G \text{tr} S(x, x) \longrightarrow G \frac{|eB|}{2\pi} \int \frac{dq_z}{2\pi} \frac{M_{\text{NJL}}(B)}{\sqrt{q_z^2 + M_{\text{NJL}}^2(B)}} f(q_z, B; \Lambda), \quad (2)$$

where $f(q_z, B; \Lambda)$ is some UV regulator function. The contact interaction couples all states in the LLL so that the Landau degeneracy factor $|eB|$ for the LLL appears. The intrinsic property of the model is that the chiral condensate has the same B -dependence as the mass gap:

$$\langle \bar{\psi}\psi \rangle_{\text{NJL}}^B \simeq -\frac{1}{G} M_{\text{NJL}}(B). \quad (3)$$

Depending on the regularization schemes, $M_{\text{NJL}}(B)$ can be $\sim |eB|^{1/2}$ (proper time regularization [15]), or $\sim \Lambda$ (four momentum cutoff [10]), or else. Each scheme has its own problems. For schemes predicting the growing behavior of the chiral condensate [16], the quark mass gap also develops as B increases. Then at finite temperature, thermal quark contributions are largely reduced so that the increasing chiral restoration temperature is naturally expected. On the other hand, if the mass gap approaches to constant, the chiral condensate also does, contradicting with the lattice results. Therefore, as far as the relation like (3) is retained, it seems that we have to abandon either the increasing chiral condensate or the reduction of critical temperatures.

This dilemma can be bypassed if we use the gluon exchange type interactions with the IR enhancement and UV suppression. To emphasize the point, we consider a simple model for the

²This suggests that even at $T = 0$, the nonperturbative gluons will be screened out at some critical value of B such that the screening mass, $m_D \sim g_s |eB|^{1/2} \sim N_c^{-1/2} |eB|^{1/2}$, becomes comparable to Λ_{QCD} (see also Sec.4).

³ For hadronic fluctuations at small $|eB|$, see Ref. [11], where chiral perturbation theory should be at work.

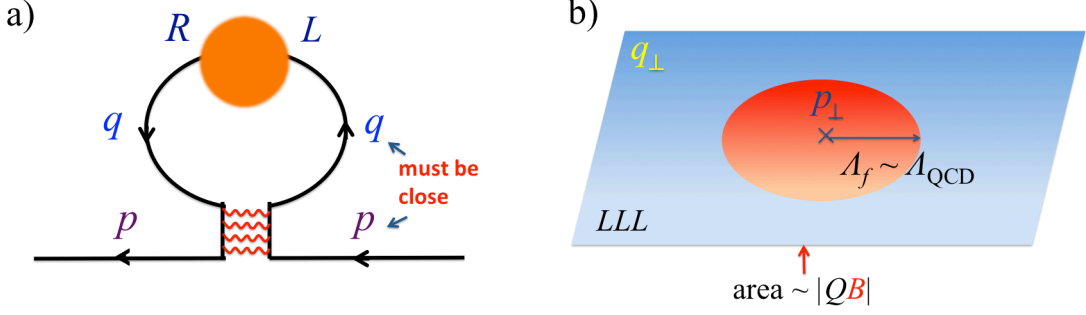


Figure 1: (a) The Schwinger-Dyson equation at large N_c for the model in Eq.(4). (b) The distribution of states in the lowest Landau level for fixed p_z . A state with momentum p_\perp can strongly couple to states within a domain of $|q_\perp - p_\perp| \leq \Lambda_f \sim \Lambda_{\text{QCD}}$.

gluon exchange with these features (for the moment we ignore spinor structures),

$$D(q) = G \theta(\Lambda_f^2 - \vec{q}^2), \quad (\Lambda_f \sim \Lambda_{\text{QCD}}) \quad (4)$$

which was proposed in [14]. In this model, the quark mass function appears to be momentum dependent. For $B \rightarrow \infty$, the Schwinger-Dyson equation at large N_c (Fig.1a) is

$$M(p; B) \simeq G \int \frac{d^4 q}{(2\pi)^4} \text{tr} S_{2\text{D}}^{LLL}(q_z) \theta(\Lambda_f^2 - |\vec{p} - \vec{q}|^2), \quad (5)$$

where B is used to reduce the quark propagator to the (1+1)-dimensional one and to separate higher Landau levels from the LLL. In contrast to the previous case, the factor $|eB|$ does not appear in front of the integral. This is because the interaction (4) does not couple all the states in the LLL, but couples the states having similar momenta (Fig.1b). This feature makes the gap B -independent. In fact, carrying out the integral over the transverse momenta, we get

$$M(p; B) \simeq \frac{G}{2\pi} \int \frac{dq_z}{2\pi} \frac{M(q; B)}{\sqrt{q_z^2 + M^2(q; B)}} \theta(\Lambda_f^2 - |p_z - q_z|^2) F(p_z - q_z), \quad (6)$$

where $F(k_z) = \sqrt{\Lambda_f^2 - |k_z|^2}$. Note that the equation does not have any explicit B dependence, so the mass gap is solely determined by the scale Λ_f , i.e. $M(p; B) = M_{\Lambda_f}(p)$.

Another important feature is the damping behavior of the mass function at large p . To see this, in (6) we take $p_z \rightarrow \infty$, for which q_z must go to ∞ . Then the integrand in the RHS goes to zero. The phase space integral cannot compensate the damping behavior of the integrand, because the integral is limited within the finite domain Λ_f around p_z . So $M(p_z) \rightarrow 0$ as $p_z \rightarrow \infty$. Thus there must be some damping region of $M(p_z)$ which we will denote Λ_χ as a function of Λ_f . The

emergence of the damping scale Λ_χ makes the chiral condensate UV finite⁴:

$$\langle \bar{\psi}\psi \rangle \sim -|eB| \int_{-\Lambda_\chi}^{\Lambda_\chi} \frac{dq_z}{2\pi} \frac{M(q_z)}{\sqrt{q_z^2 + M^2(q_z)}} \sim -|eB| F(\Lambda_f), \quad (7)$$

where $F(\Lambda_f)$ is a function of Λ_f . The chiral condensate depends on B *linearly*, as advertised.

Note that all of the discussions presented so far did not ask whether the interaction is confining or not. All we needed was the IR enhancement. Nevertheless, it is interesting to further investigate a model which not only contains properties of the IR enhancement, but also captures certain aspects of the confinement. For this purpose, we will use the model of Gribov-Zwanziger [17] which contains the linear rising potential between colored charges. Confinement is expressed as the absence of the quark continuum in the meson spectra. We will analyze the self-consistent equations for the quark self-energy and meson states at large N_c and large B , by reducing them into those of the 't Hooft model. Then we will rederive the aforementioned conclusions.

2. Dimensional reduction

We consider the Euclidean action (convention: $g_{\mu\nu} = \delta_{\mu\nu}$, $\gamma_\mu = \gamma_\mu^\dagger$):

$$S_E = \int d^4x \bar{\psi} [\not{D} + iQ\mathcal{A} + m] \psi + S_{int}, \quad (8)$$

where \mathcal{A} is a $U(1)_{em}$ gauge field, Q is a flavor matrix for electric charges, and m is the current quark mass matrix. The color gauge interaction is treated as S_{int} .

We apply an external, uniform magnetic field in spatial 3-direction⁵, which can be given by a vector potential $(\mathcal{A}_1, \dots, \mathcal{A}_4) = (0, Bx_1, 0, 0)$. For later convenience, it is useful to introduce the projection matrices. We decompose fermion fields,

$$P_\pm^Q = \frac{1 \pm \text{sgn}(QB)\sigma_3}{2}, \quad \sigma_3 = i\gamma_1\gamma_2, \quad \psi_\pm = P_\pm^Q \psi, \quad \sigma_3 \psi_\pm = \pm \text{sgn}(QB) \psi_\pm. \quad (9)$$

We expand the fermion fields in bases which diagonalize the unperturbed Lagrangian,

$$\psi_\pm(x) = \sum_{l=0} \int \frac{d^3\tilde{p}}{(2\pi)^3} \psi_\pm(l, \tilde{p}) e^{-i\tilde{p}\tilde{x}} H_l(x_1; p_2), \quad (10)$$

where $\tilde{p} \equiv (0, p_2, p_3, p_4)$. $H_l(x_1; p_2)$ is the harmonic oscillator base with $m\omega = |QB|$, whose center is located at $x_1 = p_2/QB$. If we want to write the Lagrangian as a sum of the Landau levels, we

⁴In vacuum, Λ_χ effectively plays the role of the UV cutoff in the NJL model.

⁵We try to explain only the lattice results for uniform B in the quenched QED. If we had some charged condensates (see recent discussions, [18–20]), the dynamical QED would generate vortices in electric superconductor.

should relabel $\psi_+(n, \tilde{x}) \rightarrow \psi_{n+}(\tilde{x})$, $\psi_-(n-1, \tilde{x}) \rightarrow \psi_{n-}(\tilde{x})$, $\psi_+(0, \tilde{x}) \rightarrow P_+^Q \chi(\tilde{x})$. Now the index n characterizes the Landau level. The unperturbed action becomes

$$\int d^4x \mathcal{L}_{\text{unpert.}}(x) = \int d^3\tilde{x} \left(\mathcal{L}_0(\tilde{x}) + \sum_{n=1} \mathcal{L}_n(\tilde{x}) \right), \quad (11)$$

after carrying out the integral over x_1 . The Lagrangian density is ($\gamma_L = (\gamma_3, \gamma_4)$ and $\gamma_\perp = (\gamma_1, \gamma_2)$)

$$\mathcal{L}_0 = \bar{\chi}(\tilde{x})(\not{\partial}_L + m)P_+^Q \chi(\tilde{x}), \quad \mathcal{L}_n = \bar{\psi}_n(\tilde{x}) \left(\not{\partial}_L + i \text{sgn}(QB) \sqrt{2n|QB|} \gamma_2 + m \right) \psi_n(\tilde{x}). \quad (12)$$

The propagators for Landau levels are⁶

$$\begin{aligned} \langle \chi(\tilde{x}) \bar{\chi}(\tilde{y}) \rangle &= \int \frac{d^3\tilde{p}}{(2\pi)^3} P_+^Q \frac{i}{\not{p}_L + im} e^{-i\tilde{p}(\tilde{x}-\tilde{y})}, \\ \langle \psi_n(\tilde{x}) \bar{\psi}_n(\tilde{y}) \rangle &= \int \frac{d^3\tilde{p}}{(2\pi)^3} \frac{i}{\not{p}_L - \text{sgn}(QB) \sqrt{2n|QB|} \gamma_2 + im} e^{-i\tilde{p}(\tilde{x}-\tilde{y})}. \end{aligned} \quad (13)$$

We also expand the color gauge interactions in the Ritus bases instead of usual Fourier bases [21],

$$\int d^4x \bar{\psi} \gamma_\mu t_a \psi A_{\mu a}(x) = \sum_{l, l'=0} \int \frac{d^3\tilde{p} d^3\tilde{q}}{(2\pi)^6} \bar{\psi}(l, \tilde{p}) \gamma_\mu t_a \psi(l', \tilde{q}) A_{\mu a}^{l'l'}(\tilde{p} - \tilde{q}; p_2, q_2), \quad (14)$$

where the gauge field is convoluted with the harmonic oscillator bases,

$$A_{\mu a}^{l'l'}(\tilde{p} - \tilde{q}; p_2, q_2) = \int \frac{dk_1}{2\pi} A_{\mu a}(\tilde{p} - \tilde{q}, k_1) \int dx_1 H_l(x_1; p_2) H_{l'}(x_1; q_2) e^{-ik_1 x_1}. \quad (15)$$

For later convenience, it is useful to prepare the propagator for this convoluted form,

$$\begin{aligned} &\left\langle A_{\mu a}^{l'l'}(\tilde{p} - \tilde{q}; p_2, q_2) A_{\nu b}^{j'j}(\tilde{q} - \tilde{p}; q_2, p_2) \right\rangle \\ &= \int \frac{dk_1}{2\pi} \int dx_1 \int dy_1 H_l(x_1; p_2) H_{l'}(x_1; q_2) D_{\mu\nu}^{ab}(\tilde{p} - \tilde{q}, k_1) H_{j'}(y_1; q_2) H_j(y_1; p_2) e^{-ik_1(x_1 - y_1)}, \end{aligned} \quad (16)$$

where we used $\langle A_{\mu a}(p) A_{\nu b}(q) \rangle = (2\pi)^4 \delta^4(p+q) D_{\mu\nu}^{ab}(p)$. Let us note that the $H_l(x_1; p_2)$ depends on coordinates in the combination, $x_1 - p_2/QB$. We can simultaneously shift $x_1 - p_2/QB \rightarrow x_1$, and $y_1 - p_2/QB \rightarrow y_1$, and get

$$H_l(x_1; p_2) H_{l'}(x_1; q_2) H_{j'}(y_1; q_2) H_j(y_1; p_2) \rightarrow H_l(x_1) H_{l'}(x_1; q_2 - p_2) H_{j'}(y_1; q_2 - p_2) H_j(y_1), \quad (17)$$

⁶It is convenient to attach the Ritus wavefunctions to the vertices instead of propagators, because the different Landau orbits couple only after interactions are turned on.

keeping the other part in Eq.(16) invariant. Thus the integral depends on p_2 and q_2 only through $p_2 - q_2$. This is natural consequence of the gauge invariance, $\mathcal{A}_2 = Bx_1 \rightarrow B(x_1 - c)$, that affects the origin of p_2 and q_2 but should not affect the final result.

This complicated expression can be drastically simplified if (i) the interaction shows an IR enhancement around $\sim \Lambda_{\text{QCD}}$, while it damps quickly in the UV, and (ii) the magnetic field is sufficiently strong, $|QB| \gg \Lambda_{\text{QCD}}$. The gluon propagator rapidly damps for $\tilde{p} - \tilde{q}$ and k_1 much larger than Λ_{QCD} , cutting off the domain of integral. Note also that the function $H(x_1)$ contains the Gaussian function, $\exp[-|QB|x_1^2/2]$, cutting off the domain of x_1 by $\sim 1/\sqrt{|QB|}$, as far as we consider not very high Landau orbits.

Assembling all these properties, we can make replacements at very large B ,

$$\begin{aligned} e^{-ik_1(x_1-y_1)} &\rightarrow 1, & \left(|k_1(x_1-y_1)| \sim \Lambda_{\text{QCD}}/\sqrt{|QB|} \ll 1\right) \\ H_l(x_1; p_2 - q_2) H_{l'}(x_1) &\rightarrow H_l(x_1) H_{l'}(x_1), & \left(|p_2 - q_2| \sim \Lambda_{\text{QCD}} \ll \sqrt{|QB|}\right) \end{aligned} \quad (18)$$

which simplify the Eq.(16) as

$$\left\langle A_{\mu a}^{l'j}(\tilde{p} - \tilde{q}; p_2, q_2) A_{\nu b}^{j'j}(\tilde{q} - \tilde{p}; q_2, p_2) \right\rangle \longrightarrow \int \frac{dk_1}{2\pi} \delta_{ll'} \delta_{j'j} D_{\mu\nu}^{ab}(\tilde{p} - \tilde{q}, k_1). \quad (19)$$

(Note that indices are for the orbital quanta. Whether they coincide with the Landau level indices depends on γ -matrices to which gluons couple, because γ_\perp 's flip spins.)

The final expression itself coincides with a naive expectation and is not surprising. Ultimately, as $B \rightarrow \infty$ a hopping from one Landau orbit to others (transverse dynamics) is completely suppressed. A nontrivial consequence of the asymptotic free theories, however, is that the separation of the Landau levels is achieved at a relatively small magnetic field, compared to forces like $1/p^2$. For the $1/p^2$ force, the damping of the UV force is much milder so that the LLL and higher levels do not decouple quickly. The validity of separation is quantified by examining how the replacements in Eq.(18) work.

3. The self-consistent equations at large N_c within a confining model

Now we consider the self-consistent equations, the Schwinger-Dyson and Bethe-Salpeter equations. Below we set $m = 0$. We use the gluon propagator of the Gribov-Zwanziger type [17],

$$D_{\mu\nu}^{ab}(k) = \frac{\delta_{ab}}{C_F} D_{\mu\nu}(k), \quad D_{44}(k) = -\delta_{44} \frac{8\pi\sigma}{(\vec{k}^2)^2}, \quad D_{4j}(k) = D_{ij}(k) = 0. \quad (20)$$

which is motivated by Coloumb gauge studies. D_{44} gives a linear rising potential for the color charges, and we dropped off terms without the strong IR enhancement. σ is a string tension of $O(\Lambda_{\text{QCD}}^2)$, $C_F = (N_c^2 - 1)/2N_c$ is Casimir for the adjoint representation.

We will assume that the self-energies are diagonal for each Landau level, and are translational invariant. Then the Schwinger-Dyson equation, after applying our approximations (19), becomes

$$(\not{Y}_L + \not{Y}_\perp + \Sigma_m)(n, \tilde{p}) = \int \frac{d^3 \tilde{q}}{(2\pi)^3} \left[i\gamma_4 S(n, \tilde{q}; \Sigma) i\gamma_4 \right] \int \frac{dk_1}{2\pi} D_{44}(\tilde{p} - \tilde{q}, k_1), \quad (21)$$

that is diagonal for the Landau orbits.

The computations are particularly simple for the LLL. Below we drop the subscript 0 for the LLL. The matrix γ_\perp drops off because of the projection operator P_+^Q , so that RHS of the equation does not have γ_\perp , meaning that $\Sigma_\perp(\tilde{p}) = 0$ in LHS. Then the equation looks like

$$(\not{Y}_L + \Sigma_m)(\tilde{p}) = \int \frac{d^3 \tilde{q}}{(2\pi)^3} i\gamma_4 \frac{i[\not{q}_L - \not{Y}_L(\tilde{q})] + \Sigma_m(\tilde{q})}{[q_L - \Sigma_L(\tilde{q})]^2 + \Sigma_m^2(\tilde{q})} i\gamma_4 \int \frac{dk_1}{2\pi} D_{44}(\tilde{p} - \tilde{q}, k_1). \quad (22)$$

Recall that the quark transverse momentum is gauge dependent in the sense that $A_2 = Bx_1 \rightarrow B(x_1 - c)$ affects the origin of the transverse momenta as $p_2 \rightarrow p_2 + QBc$ and $q_2 \rightarrow q_2 + QBc$. Doing this shift, the gluon propagator is unaffected while the modification appears only through the self-energy as $\Sigma(p_L, p_2) \rightarrow \Sigma(p_L, p_2 + QBc)$, and $\Sigma(p_L, q_2) \rightarrow \Sigma(p_L, q_2 + QBc)$. This replacement does not change the structure of the self-consistent equation at all, so that we obtain the same solutions for the two cases, $\Sigma(p_L, p_2) = \Sigma(p_L, p_2 + QBc)$, which means that $\Sigma(p)$ is p_2 independent.

As a consequence, we can factorize the equation:

$$(\not{Y}_L + \Sigma_m)(p_L) = \int \frac{d^2 q_L}{(2\pi)^2} i\gamma_4 \frac{i[\not{q}_L - \not{Y}_L(q_L)] + \Sigma_m(q_L)}{[q_L - \Sigma_L(q_L)]^2 + \Sigma_m^2(q_L)} i\gamma_4 \int \frac{dk_1 dq_2}{(2\pi)^2} D_{44}(\tilde{p} - \tilde{q}, k_1). \quad (23)$$

The gluon propagator is smeared by integrating out the transverse momentum,

$$D_{44}^{2D}(p_z - q_z) = \int \frac{dk_1 dq_2}{(2\pi)^2} D_{44}(\tilde{p} - \tilde{q}, k_1) = \int \frac{d^2 \vec{k}_\perp}{4\pi^2} \frac{-8\pi\sigma}{((p_z - q_z)^2 + \vec{k}_\perp^2)^2} = -\frac{2\sigma}{(p_z - q_z)^2}, \quad (24)$$

which is a confining propagator in (1+1) dimensions. The form of the Schwinger-Dyson equation is exactly same as that of 't Hooft model [22] in axial gauge, $A_z = 0$, whose solution is known. The string tension is related to the two dimensional gauge coupling as $N_c g_{2D}^2 = 4\sigma$.

The magnetic field disappears from the equation. The only role of the magnetic field is to make the unperturbed quark propagator (1+1)-dimensional and to separate the LLL from the other Landau orbits. The only scale in the equation is $\sigma \sim \Lambda_{\text{QCD}}^2$. Using the dressed (1+1)-dimensional quark propagator for Eq.(7), we can get the chiral condensate at finite B .

Since the quark self-energies are not always well-defined, we will also consider the meson states which are free from any ambiguities. One can estimate an effective or constituent quark mass by examining the meson spectra. To do this, we study the Bethe-Salpeter equation.

We shall move to Minkowski space, and treat the homogenous Bethe-Salpeter equation assuming that the total momentum of a quark and an anti-quark is sufficiently close to the pole. The equation for a color singlet channel is

$$\Psi(P; q)_{\alpha\beta}^{ff'} = - \int \frac{d^4 k}{(2\pi)^4} [S(k - P) \gamma_0 \Psi(P; k) \gamma_0 S(k + P)]_{\alpha\beta}^{ff'} D_{00}(k - q), \quad (25)$$

where $\Psi(P; q)$ is a meson wavefunction with total momentum P and relative momentum q for a quark and an anti-quark. The indices α, β (f, f') are for spinor (flavor) indices. We can decompose the wavefunction into different spinor combinations and flavor structures,

$$\Psi_{\alpha\beta}^{ff'} = (\Psi_S + \Psi_5 \gamma_5 + \Psi_L \gamma_L + \dots)_{\alpha\beta} \mathbf{1}_{ff'} + \dots. \quad (26)$$

We must apply further decompositions or take appropriate linear combinations, because a magnetic field breaks isospin and rotational symmetries explicitly. Below we restrict ourselves to the meson states with electric charges and spinor combinations for which we can close the equation only by the LLL. One can select out such mesons using the projection operator $P_+^Q = (1 + \text{sgn}(QB)\gamma_1\gamma_2)/2$. For the two flavor case, the following structures satisfy the condition,

$$(u\bar{u}, d\bar{d}) \otimes (1, \gamma_5, \gamma_L, \gamma_L\gamma_5, \sigma_{LL'}, \sigma_{\perp\perp'}), \quad (u\bar{d}, d\bar{u}) \otimes (\gamma_\perp, \gamma_\perp\gamma_5, \sigma_{L\perp}). \quad (27)$$

For instance, neutral pions remain light, while charged ones acquire the masses $\sim \sqrt{|eB|}$. Other examples are vector mesons. The longitudinal (transverse) component of neutral (charged) vector mesons can remain light, while others not⁷. This observation seems to be consistent with recent lattice results [19] and model calculations [23].

Below we consider the neutral scalar component, Ψ_S , as an illustration, taking only the LLL into account. As we saw, the dressed quark propagator is independent of the transverse momentum. Then we can conclude that $\Psi_S(P; q)$ is independent of P_\perp , because P_\perp -dependence appears only through Ψ_S so that different P_\perp gives the same equation and thereby the same solution. We can also conclude that $\Psi_S(P; q)$ is independent of q_\perp , because simultaneous shift of momenta, $q_\perp \rightarrow q_\perp + c$ and $k_\perp \rightarrow k_\perp + c$, does not affect the equation.

Therefore we can again factorize the equation,

$$\Psi_S(P_L; q_L) = - \int \frac{d^2 k_L}{(2\pi)^2} [S(k_L - P_L) \gamma_0 \Psi_S(P_L; k_L) \gamma_0 S(k_L + P_L)] \int \frac{d^2 k_\perp}{(2\pi)^2} D_{00}(k - q), \quad (28)$$

⁷More careful considerations are necessary beyond the large N_c limit where the annihilation diagrams may contribute. In this respect studies of instantons in a magnetic field are important [24].

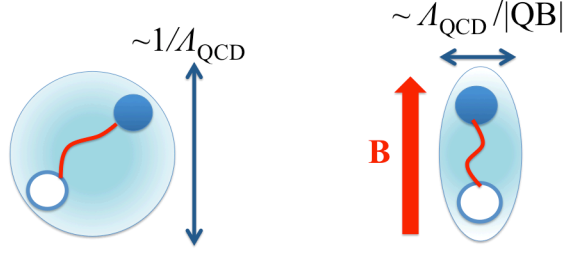


Figure 2: A structure of mesons listed in Eq.(27). With a magnetic field, a quark and an anti-quark aligns along direction of the magnetic field. The meson can move in the transverse direction without costing much energy.

and obtain the Bethe-Salpeter equation in the 't Hooft model. There is no B -dependence in the equation, so masses of mesons are characterized by current quark masses and Λ_{QCD} . There are an infinite tower of meson spectra, due to the string oscillations along z -direction.

The total momentum of a quark and an anti-quark is independent of transverse momentum, meaning that the meson can move freely in the transverse direction without costing energy⁸. The inclusion of the higher Landau level can generate the dependence on the transverse momentum, which is suppressed by a factor of $\Lambda_{\text{QCD}}/|eB|^{1/2}$. This leads to a spatially anisotropic meson Lagrangian [8] for which mesonic fluctuations are strong due to enhanced phase space in the infrared. It may significantly affect dynamics at finite temperature [25], like a system close to the quasi-long range order⁹ [27].

On the other hand, the independence of the relative momentum means that the internal meson wavefunction behaves as

$$\Psi_S^{\text{rel.}}(\vec{r}) = \int dq_z d^2\vec{q}_\perp \Psi_S^{\text{rel.}}(q_z) e^{i\vec{q}\cdot\vec{r}} \sim \delta^2(\vec{r}_\perp) \psi_S(r_z), \quad (29)$$

that is, a quark and an anti-quark align along the z -direction, and the inter-particle distance is $\sim 1/\Lambda_{\text{QCD}}$ (Fig.3). Couplings with the higher Landau levels will introduce the width for the transverse wavefunction, and the mean square radius should be $\sqrt{\langle \vec{r}_\perp^2 \rangle} \sim \Lambda_{\text{QCD}}/|QB|$.

4. Conclusion

We have discussed that the quark mass gap in the presence of a magnetic field stays around $\sim \Lambda_{\text{QCD}}$ or less, provided that it is mainly generated by the nonperturbative part of the gluon exchange. Accordingly the chiral condensate grows at most linearly as a function of B . This

⁸The meson dynamics is effectively (1+1)-dimensional, in the sense that the transverse momenta do not affect much the energy dispersion. This means that mesons can easily propagate in the transverse directions.

⁹ In the strict quasi-long range order, $\langle \bar{\psi}\psi \rangle \sim \rho \langle e^{i\theta} \rangle \rightarrow 0$ due to the IR divergence of the phase fluctuations. But the quark mass gap can remain finite as far as $\rho \neq 0$ [26].

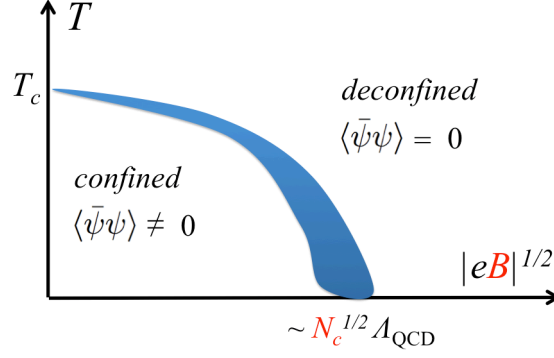


Figure 3: Our expectation for the phase diagram in $|eB|^{1/2} - T$ plane. At $T = 0$, a magnetic field would reach its critical strength around $|eB|^{1/2} \sim N_c^{1/2} \Lambda_{\text{QCD}}$.

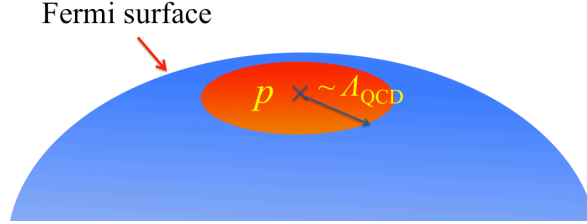


Figure 4: The momentum space structure near the Fermi surface. If the curvature is small, the structure of the Schwinger-Dyson equation for a non-uniform chiral condensate is very similar to that for the LLL in a magnetized system (See also Fig.1b).

tendency seems to match with the current lattice data. But to complete our arguments, more detailed discussions about the UV tail of the gluon propagator are necessary. We leave it for future studies.

We have not explicitly taken into account the screening effects for gluons by taking large N_c . Including $1/N_c$ corrections, the screening effects should grow as B increases, reducing the nonperturbative forces. Accordingly the growth rate of the chiral condensate should become smaller than the large N_c estimate. We expect that for an extremely strong magnetic field, the screening effects will exceed some critical strength to make the system deconfined (Fig.3). For $T = 0$, the critical B is roughly estimated as $|eB|^{1/2} \sim N_c^{1/2} \Lambda_{\text{QCD}}$, by equating the numbers of virtual quark and gluon excitations at low energies, $N_c |eB| \Lambda_{\text{QCD}} \sim N_c^2 \Lambda_{\text{QCD}}^3$.

This situation seems to be quite analogous to what would happen in cold, dense quark matter. At large quark chemical potential μ_q , the area of the quark Fermi surface is enhanced. Accordingly the phase space for quark excitations at low energy increases as $\sim N_c \mu_q^2 \Lambda_{\text{QCD}}$, so that the screening

effects for gluons become stronger¹⁰. The critical chemical potential for deconfinement has been estimated to be $\mu_q \sim N_c^{1/2} \Lambda_{\text{QCD}}$ [29]. Detailed quantitative estimates crucially depend upon the existence of the quark mass gap near the Fermi surface. The gap of $O(\Lambda_{\text{QCD}})$ can emerge if we take into account the non-uniform chiral condensates. The corresponding Schwinger-Dyson equation [14] takes very similar form as that in a magnetized system, Eq.(6). This correspondence becomes better at larger μ_q , for which the curvature of the Fermi surface is small enough (Fig.4).

In summary, the lattice Monte Carlo simulation at finite B is a promising tool to extract out quantitative information about the gluon polarizations. Such information has great relevance to understanding dense quark matter, or more concretely, physics near the Fermi surface. On the other hand, information about the bulk Fermi sea should be supplemented by the lattice studies for the two-color QCD (or isospin QCD) at large quark (isospin) density without a magnetic field [30]. These discussions will be expanded elsewhere.

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¹⁰The phase space enhancement by μ_q strongly depends on dimensions. In particular, in (1+1) dimensions, the screening does not get stronger. For QCD in (1+1)-dimensions, the system is always confined [28].

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